ALGEBRA STRUCTURE
(Group of Permutation)

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1. Group of Permutation
   - Group of Permutations
   - Symmetric Group

2. Orbit and Cycle
   - Orbit
   - Cycle
   - Alternating Group
Outline

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Group of Permutations

**Definition**

Permutation on a set $A$ is a one-to-one function from $A$ onto itself.

**Construction**

Let $A$ be a non-empty set and $S_A$ be the set of all permutations in $A$. Then $S_A$ is a group under permutation multiplication.
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Symmetric Group

Definition
Let \( A = \{1, 2, 3, \ldots, n\} \), then group of all permutations on \( A \) is called **symmetric group** \( n \), and be denoted as \( S_n \).

Example
Let \( A = \{1, 2, 3\} \) then \( S_3 \) has \( 3! = 6 \) elements. All permutations on \( A \) can be described below.

\[
\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},
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\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},
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\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},
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\rho_2 = \begin{pmatrix}
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\]
Let $\sigma$ be a permutation of a set $A$. The equivalence classes determined by the equivalence relation

$$a \sim b \iff b = \sigma^n(a)$$

are the **orbits** of $\sigma$.

The orbits of permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 6 & 3 & 7 & 4 & 5 & 2 \end{pmatrix}$$

of $S_8$ can be found by applying $\sigma$ repeatedly, obtaining symbolically

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$$1 \rightarrow 8 \rightarrow 2 \rightarrow 1$$
A permutation $\sigma \in S_n$ is a **cycle** if $\sigma$ has at most one orbit containing more than one element. The **length** of a cycle is the number of elements in its largest orbit.

**Note**

Unlike in the orbit notation, the order of elements in the cycle notation determines moving flow. For example, $(1, 8, 2) = (8, 2, 1) = (2, 1, 8)$ but $(1, 8, 2) \neq (1, 2, 8)$.

**Theorem**

Each permutation $\sigma$ in a finite set is a product of disjoint cycles.
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Each permutation $\sigma$ in a finite set is a product of disjoint cycles.
Transposition

**Definition**
A cycle of length 2 is called transposition.

Each cycle can be described as a product of transpositions,

\[(a_1, a_2, a_3, \cdots, a_{n-1}, a_n) = (a_1, a_n)(a_1, a_{n-1}) \cdots (a_1, a_3)(a_1, a_2)\]

Therefore, a permutation is also a product of transpositions.

**Theorem**
Let \(\sigma \in S_n\) and \(\tau\) be a transposition on \(S_n\). The number of orbits of \(\sigma\) and the number of orbits of \(\tau\sigma\) differ by 1.
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Even and Odd Permutation

A permutation of a finite set is **even** or **odd** according to whether it can be expressed as a product of an even number of transposition or the product of an odd number of transposition, respectively.

Alternating Group

The subgroup of $S_n$ consisting even permutations is called **alternating group**, $A_n$ on $n$ letters.
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